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The Toda flows preserving small cells of the flag variety G/B and Kazhdan's x_0-grading

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Abstract

There exist many works on full Kostant–Toda flows whose initial points belong to a grand cell of the flag variety. In this paper we study the full Kostant–Toda flows whose initial points belong to small cells other than the grand cell and preserving small cells. To obtain such flows, we consider the cell decomposition of the grand cell induced by Bruhat decomposition of the flag variety. We show that such full Kostant–Toda flow can be reduced to the ordinary tridiagonal Toda flow by using sheaf theoretical methods.

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1. Introduction

The Lax operator of the *full Kostant–Toda lattice* is the Hessenberg matrix of the form

$$L(t) = \Lambda + \sum_{1\le j\le i\le n} L_{i,j}(t)E_{i,j}, \tag{1.1}$$

where $E_{i,j}$ is the i, j matrix unit and $\Lambda = \sum_{j=1}^{n-1} E_{j,j+1}$, which satisfies the equation

$$\dot{L}(t) = [L(t)_+, L(t)]. \tag{1.2}$$

We consider the reduction of the full Kostant–Toda lattice to the ordinary Toda lattice. We know that the full Kostant–Toda lattice is itself integrable by virtue of many works: [3,5,8,11] etc. Let G be $GL(n, \mathbf{C})$. Let $B \subset G$ be the Borel subgroup of upper triangular matrices and $N \subset B$ be the subgroup whose diagonal components are all 1. Moreover $\bar{B}$ and $\bar{N}$ are opposites of B and N respectively. Let $\mathfrak{b}$, $\mathfrak{n}$, $\bar{\mathfrak{b}}$ and $\bar{\mathfrak{n}}$ be Lie algebras of B, N, $\bar{B}$ and $\bar{N}$ respectively. To obtain the solution of the Lax equation, we consider the Gauss decomposition

$$W_\infty(t)^{-1} W_0(t) = \mathrm{e}^{tL}, \tag{1.3}$$

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where $W_\infty(t) \in \bar{N}$, $W_0(t) \in B$ and L is the constant matrix of the form $\Lambda + \bar{\mathfrak{b}}$. Put $L(t) = W_\infty(t) L W_\infty(t)^{-1}$. Then $L(t)$ satisfies (1.2) and has the form $\Lambda + \bar{\mathfrak{b}}$. It is natural to regard e^{tL} of (1.3) as a point of the flag variety of G/B rather than an element of G. As we know, the flag variety G/B has the Bruhat decomposition [1]

$$G/B = \sqcup_{\sigma \in \mathcal{S}_n} \bar{N} \sigma B/B, \tag{1.4}$$

where $\mathcal{S}_n$ is the n-th symmetric group. We see that e^{tL} belongs to the ϕ-cell $\bar{N}B/B$ from the Gauss decomposition (1.3). Then it is natural to consider what equation would be induced if e^{tL} belonged to the σ-cell $\bar{N}\sigma B/B$. In other words, we want to know the equation which $L(t) = W_\infty(t) L W_\infty(t)^{-1}$ satisfies, where $W_\infty(t) \in \bar{N}$ satisfies the Bruhat decomposition

$$W_\infty(t)^{-1} \sigma W_0(t) = e^{tL}. \tag{1.5}$$

In [7], the authors efficiently explain the Painlevé analysis (or blow-ups of solutions) of the Toda lattice (cf. [4,6]) via the topology of the iso-level set of the Toda lattice (flag variety; cf. [2,15,17]). If e^{tL} leaves the ϕ-cell for another cell at $t = t_0$, $L(t) = W_\infty(t) L W_\infty(t)^{-1}$ has a pole at $t = t_0$, where $W_\infty(t)$ satisfies the Gauss decomposition

$$W_\infty(t)^{-1} W_0(t) = e^{tL}, \quad (t \in (t_0 - \epsilon, t_0 + \epsilon) - \{t_0\} \text{ for } 0 < \epsilon << 1).$$

They determine the cells from which e^{tL} leaves for the singularity of the Toda lattice at $t = t_0$. In this paper we focus on the Toda flows whose initial points belong to the σ ($\neq \phi$) cell of the flag variety and preserving the σ-cell. We realize such flows by using Gauss decompositions equivalent to (1.5). Furthermore we exhibit a simple example in the Appendix which explains that the flows start with the initial point of a small cell and the reason why the flows with the initial point of the ϕ-cell leave for the other cell in finite time [7]. To obtain the dynamical system, we have the following Gauss decompositions of two types equivalent to (1.5):

$$\tilde{W}_\infty(t)^{-1} \tilde{W}_0(t) = \sigma^{-1} e^{tL}, \tag{1.6}$$

where $\tilde{W}_\infty(t) \in \bar{N} \cap \sigma^{-1} \bar{N} \sigma$ and $\tilde{W}_0(t) \in B$ and

$$\tilde{W}_\infty(t)^{-1} \tilde{W}_0(t) = e^{tL} \sigma^{-1}, \tag{1.7}$$

where $\tilde{W}_\infty(t) \in \bar{N}$ and $\tilde{W}_0(t) \in B \cap \sigma B \sigma^{-1}$. From the topological point of view, the decomposition of (1.6) relates to a cell decomposition of the ϕ-cell of the flag variety such as

$$\bar{N}B/B = \sqcup_{\sigma \in \mathcal{S}_n} Y_\sigma, \tag{1.8}$$

where Y_σ is an open dense set of $(\bar{N} \cap \sigma^{-1} \bar{N} \sigma) B/B$. On the other hand the decomposition of (1.7) relates to the decomposition

$$\bar{N}B = \sqcup_{\sigma \in \mathcal{S}_n} Y'_\sigma, \tag{1.9}$$

where Y'_σ is an open dense set of $\bar{N}(B \cap \sigma B \sigma^{-1})$. We give precise definitions of Y_σ and Y'_σ in Sections 2 and 4.

In Sections 3 and 4, we consider $\tilde{L}(t) = \tilde{W}_\infty(t) L \tilde{W}_\infty(t)^{-1}$, where $\tilde{W}_\infty(t)$ satisfies (1.7). We see that $\tilde{L}(t)$ satisfies the ordinary Lax equation $\dot{\tilde{L}}(t) = [\tilde{L}(t)_+, \tilde{L}(t)]$ and $\tilde{L}(t)$ has the form $\Lambda + \bar{\mathfrak{b}}$. If $\tilde{L}(t)$ is a Jacobi element we call $\tilde{L}(t)$ a Lax operator of the *Toda lattice*. We grasp that the full Kostant–Toda lattice is the Hamiltonian system on the affine coordinate ring of Hessenberg matrices. We describe the full Kostant–Toda lattice in sheaf theoretical language from the point of view of the Hamiltonian formalism. We apply our formulation to solve the following problem.

Problem. "Characterize the constant L of the form $\Lambda + \bar{\mathfrak{b}}$ for which $\tilde{L}(t) = \tilde{W}_\infty(t) L \tilde{W}_\infty(t)^{-1}$, where $\tilde{W}_\infty(t)$ satisfies the decomposition (1.7), becomes the Lax operator of the Toda lattice."

It is clear that $\tilde{L}(t)$ is the Lax operator of the full Kostant–Toda lattice for any L of the form $\Lambda + \bar{\mathfrak{b}}$. The problem is finding the conditions for L so that $\tilde{L}(t)$ is a Jacobi element. In the case of $\sigma = id$, it is enough for L to be a Jacobi element, from the following proposition.

Proposition 1.1. *Let $L \in \Lambda + \bar{\mathfrak{b}}$ be a Jacobi element. Then $L(t) = W_\infty(t) L W_\infty(t)^{-1}$ satisfies the Lax equation (1.2), $L(t)$ is a Jacobi element and $L(0) = L$, where $W_\infty(t) \in \bar{N}$ and $W_0(t) \in B$ satisfies the Gauss decomposition $W_\infty(t)^{-1} W_0(t) = \mathrm{e}^{tL}$.*

This is straightforward from Th.2.4 of [13]. However in the case of $\sigma \neq id$, we cannot apply the method of Kostant. The set of $n \times n$ complex Hessenberg matrices has foliation where each leaf, S_m, is the iso-level set of the full Kostant–Toda lattice which is defined in Section 4 precisely. Then the sheaf theoretical language is useful for restricting the results of the above problem to each of them. In the argument of Section 4, we see that the wave operator of the full Kostant–Toda flow which starts from a point of the small cell of the flag variety has a pole at $t = t_0$. Since the flow on the small cell meets the grand cell at $t = t_0$, this singularity occurs. Note that the singularity of the Toda flows which start from points of the ϕ-cell have singularities when they meet small cells [7]. In [9], we consider the cell preserving flows which do not meet other cells. In [16], the singularities of the full Kostant–Toda flow with respect to k-chop integrals are considered. The singularities are classified as type I and type II. As regards to our work, the singularities in this paper would relate to singularities of 0-chop flow of type II. However we do not assume the coincidence of eigenvalues of the initial point. Thus the relation between our singularities and those of type II is unclear.

2. The Gauss decomposition corresponding to the Bruhat decomposition of G/B and the associated full Kostant–Toda lattice

Put $X_\sigma = \bar{N}\sigma B/B$ and $X_\phi = \bar{N}B/B$ in (1.4), where we identify σ with $\sum_{i=1}^n E_{\sigma(i),i}$. The ordinary Gauss decomposition of the full Kostant–Toda lattice means that e^{tX} belongs to the ϕ-cell. However the Bruhat decompositions for other cells differ from the Gauss decomposition. In this section we use the isomorphism of varieties of cells of the Bruhat decomposition ([1], p. 193, Th.(a)) for the decomposition of X_ϕ. This decomposition gives the Lax type equations (which differ from the ordinary Lax equation subtly) and parameterization by $\mathcal{S}_n$. For $\sigma \in \mathcal{S}_n$, we define the homeomorphism $m_\sigma : G \to G$ by $m_\sigma(g) = \sigma^{-1} g$. Then we see that

$$\bar{N}\sigma B \cong m_\sigma(\bar{N}\sigma B) = \sigma^{-1}(\bar{N}\sigma B) = (\sigma^{-1}\bar{N}\sigma)B.$$

We move the cell $\bar{N}\sigma B$ by m_σ and consider a subset of $\bar{N}B/B$ such as $(\bar{N}B \cap \sigma^{-1}\bar{N}\sigma B)/B$. We will show that X_ϕ is decomposed into cells associated with $(\bar{N}B \cap \sigma^{-1}\bar{N}\sigma B)/B$.

Proposition 2.1. *It holds that*

$$\bar{N}B \cap \sigma^{-1}\bar{N}\sigma B = (\bar{N} \cap \sigma^{-1}\bar{N}\sigma)B.$$

Proof. The inclusion $(\bar{N} \cap \sigma^{-1}\bar{N}\sigma)B \subset \bar{N}B \cap \sigma^{-1}\bar{N}\sigma B$ is clear. We show the converse. Suppose $x \in \bar{N}B \cap \sigma^{-1}\bar{N}\sigma B$. Put $x = \sigma^{-1} n \sigma b = n' b'$, where $n, n' \in \bar{N}$ and $b, b' \in B$.

Lemma 2.2. *It holds that*

$$\sigma^{-1}\bar{N}\sigma \subset (\bar{N} \cap \sigma^{-1}\bar{N}\sigma)B,$$

for any $\sigma \in \mathcal{S}_n$.

Proof. Suppose $a = \sum_{i,j=1}^n a_{i,j} E_{i,j} \in \bar{N}$; of course $a_{i,i} = 1$ and $a_{i,j} = 0$ for $i < j$. We see that

$$\sigma^{-1} a \sigma = \sum_{i,j=1}^n a_{i,j} E_{\sigma^{-1}(i),\sigma^{-1}(j)}.$$

Suppose $k > \ell$ and $\sigma^{-1}(k) < \sigma^{-1}(\ell)$. Put $b = 1 - a_{k,\ell} E_{k,\ell}$. Then we see that $b \in \bar{N}$ and $\sigma^{-1} b \sigma \in N \subset B$. Moreover the k, ℓ component of ab is 0. More generally we have the following lemma.

Lemma 2.3. *For $a \in \bar{N}$, there exist $b_1, \ldots, b_r \in \bar{N}$, such that the i, j components of $ab_1 \cdots b_r$ where $i > j$ and $\sigma^{-1}(i) < \sigma^{-1}(j)$ are 0 and $\sigma^{-1} b_i \sigma \in N \subset B$, $i = 1, \ldots, r$.*

Proof. Let ℓ_0 be the minimal number in $\{1,\ldots,n\}$ where there exists $k \in \{1,\ldots,n\}$ such that $k > \ell$ and $\sigma^{-1}(k) < \sigma^{-1}(\ell)$. Let k_0 be the minimal number in $\{1,\ldots,n\}$ satisfying $k > \ell_0$ and $\sigma^{-1}(k) < \sigma^{-1}(\ell_0)$. Put $b(k_0,\ell_0) = 1 - a_{k_0,\ell_0}E_{k_0,\ell_0} \in \bar{N}$. As we saw above, we see that the k_0, ℓ_0 component of $ab(k_0,\ell_0)$ is 0 and $\sigma^{-1}b(k_0,\ell_0)\sigma \in N \subset B$. Put $a^{(1)} = ab(k_0,\ell_0)$. Let k_1 be the minimal number next to k_0 satisfying $k > \ell_0$ and $\sigma^{-1}(k) < \sigma^{-1}(\ell_0)$. Put $b(k_1,\ell_0) = 1 - a^{(1)}_{k_1,\ell_0}E_{k_1,\ell_0} \in \bar{N}$. Then we see that the k_1, ℓ_0 component of $a^{(1)}b(k_1,\ell_0)$ is 0 and $\sigma^{-1}b(k_1,\ell_0)\sigma \in N \subset B$. Moreover we see that the right multiplication of $b(k_1,\ell_0)$ is the addition of the k_1-th column of $a^{(1)}$ multiplied by $-a^{(1)}_{k_1,\ell_0}$ to the ℓ_0-th column of $a^{(1)}$. Since $k_1 > k_0$, the k_0-th component of the k_1-th column of $a^{(1)}$ is 0. Then we see that the k_0, ℓ_0 component of $a^{(1)}b(k_1,\ell_0)$ is still 0. We repeat these manipulations until the numbers such as $k > \ell_0$ and $\sigma^{-1}(k) < \sigma^{-1}(\ell_0)$ are exhausted. Let $a^{(r_1)}$ be the resulting matrix of these manipulations. Let ℓ_1 be the minimal number next to ℓ_0 such that there exist $k \in \{1,\ldots,n\}$ such that $k > \ell$ and $\sigma^{-1}(k) < \sigma^{-1}(\ell)$. We also proceed with the same manipulation as before to the ℓ_1-th column of $a^{(r_1)}$. Since $\ell_1 > \ell_0$, the manipulation for the ℓ_1-th column has no influence on the ℓ_0-th column. Then we obtain $b_1,\ldots,b_r \in \bar{N}$ of this lemma. □

For $a \in \bar{N}$, there exist $b_1,\ldots,b_r \in \bar{N}$ of Lemma 2.3. Then we see that $\sigma^{-1}ab_1\cdots b_r\sigma \in \bar{N}$. Indeed put $\tilde{a} = ab_1\cdots b_r$. Then we see that $\tilde{a} \in \bar{N}$ and $\sigma^{-1}\tilde{a}\sigma = \sum_{k,\ell}\tilde{a}_{k,\ell}E_{\sigma^{-1}(k),\sigma^{-1}(\ell)}$. We have $\tilde{a}_{k,\ell} = 0 \Leftrightarrow k > \ell$ and $\sigma^{-1}(k) < \sigma^{-1}(\ell)$ from Lemma 2.3. Then we have $\sigma^{-1}\tilde{a}\sigma \in \bar{N}$. Then we see that $\sigma^{-1}\tilde{a}\sigma \in \bar{N} \cap \sigma^{-1}\bar{N}\sigma$. Since $a = \tilde{a}b_1^{-1}\cdots b_r^{-1}$, we have

$$\sigma^{-1}a\sigma = \sigma^{-1}\tilde{a}\sigma(\sigma^{-1}b_1^{-1}\sigma)\cdots(\sigma^{-1}b_r^{-1}\sigma) \in (\bar{N} \cap \sigma^{-1}\bar{N}\sigma)B.$$

Since $x = \sigma^{-1}n\sigma b = n'b'$ and $\sigma^{-1}n\sigma = \tilde{n}\tilde{b}$, where $\tilde{n} \in \bar{N} \cap \sigma^{-1}\bar{N}\sigma$ and $\tilde{b} \in B$, by the uniqueness of the Gauss decomposition, we have $n' = \tilde{n}$ and $b' = \tilde{b}b$. Then we have $x \in (\bar{N} \cap \sigma^{-1}\bar{N}\sigma)B$. This completes the proof of Lemma 2.2 and we have $\bar{N}B \cap \sigma^{-1}\bar{N}\sigma B \subset (\bar{N} \cap \sigma^{-1}\bar{N}\sigma)B$. This completes the proof of Proposition 2.1. □

Put $\pi_\sigma = \{[nb] \in \bar{N}B/B \mid n_{i,j} = 0, i > j, \sigma^{-1}(i) < \sigma^{-1}(j)\}$. Then we see that X_ϕ is homeomorphic to $\mathbf{C}^{n(n-1)/2}$. We define the subsets of the ϕ-cell $\bar{N}B/B$ associated with $\sigma \in \mathcal{S}_n$ by $Y_\sigma = \pi_\sigma - \cup_{\tau>\sigma}\pi_\tau$. Then we see that $Y_{\sigma_0} = \{0\}$, where σ_0 is the longest element in $\mathcal{S}_n$ with respect to Bruhat order.

Proposition 2.4. *There exists a cell decomposition of $\bar{N}B/B$ such as*

$$\bar{N}B/B = \sqcup_{\sigma\in\mathcal{S}_n} Y_\sigma. \tag{2.1}$$

Then each cell satisfies

$$\bar{Y}_\sigma = \sqcup_{\tau\geq\sigma} Y_\tau,$$

where $\bar{Y}_\sigma$ is the closure in π_σ with respect to the relative topology of π_σ induced from $\bar{N}B/B$ and $\geq$ means the Bruhat order.

Proof. First, we show that the union of (2.1) is indeed disjoint. Assume that we have neither $\sigma \leq \tau$ nor $\sigma \geq \tau$. We see that $\pi_\sigma \cap \pi_\tau \subset (\cup_{\iota>\sigma}\pi_\iota) \cap (\cup_{\iota>\tau}\pi_\iota)$. Since $Y_\sigma = \pi_\sigma - \cup_{\iota>\sigma}\pi_\iota$ and $Y_\tau = \pi_\tau - \cup_{\iota>\tau}\pi_\iota$, we have $Y_\sigma \cap Y_\tau = \phi$. If $\sigma > \tau$, we can show $Y_\sigma \cap Y_\tau = \phi$ similarly. We see that $\bar{Y}_{\sigma_0} = Y_{\sigma_0}$. We assume that the statement of the proposition is true for $\tau > \sigma$. Then we have

$$\bar{Y}_\sigma = \pi_\sigma = Y_\sigma \sqcup (\pi_\sigma - Y_\sigma) = Y_\sigma \sqcup \cup_{\tau>\sigma}\pi_\tau = Y_\sigma \sqcup \cup_{\tau>\sigma}\bar{Y}_\tau = Y_\sigma \sqcup \cup_{\tau>\sigma}\sqcup_{\iota\geq\tau}Y_\iota = \sqcup_{\tau\geq\sigma}Y_\tau. \quad \square$$

Y_σ is an open dense subset of $(\bar{N} \cap \sigma^{-1}\bar{N}\sigma)B/B$ with respect to the relative topology. Suppose $\mathrm{e}^{tL} \in \bar{N}\sigma B/B$. Then we have the Bruhat decomposition

$$W_\infty(t)^{-1}\sigma W_0(t) = \mathrm{e}^{tL}, \tag{2.2}$$

where $W_\infty(t) \in \bar{N}$ and $W_0(t) \in B$ and $L \in \Lambda + \bar{\mathfrak{b}}$ is the constant matrix. We can imply the Gauss decomposition of (2.1) as follows. From the proof of Lemma 2.3, we see that there exists $n(t) \in \bar{N}$ such that

$$\sigma^{-1}W_\infty(t)^{-1}n(t)\sigma \in \bar{N} \quad \text{and} \quad \sigma^{-1}n(t)\sigma \in N \subset B.$$

Put $\tilde{W}_\infty(t) = n(t)^{-1} W_\infty(t)$. Then we have $\tilde{W}_\infty(t) \in \bar{N} \cap \sigma \bar{N} \sigma^{-1}$. We have

$$(\sigma^{-1} \tilde{W}_\infty(t)\sigma)^{-1}\{(\sigma^{-1} n(t)^{-1}\sigma) W_0(t)\} = \sigma^{-1} \mathrm{e}^{tL} \tag{2.3}$$

from (2.2). Note that $\sigma^{-1} \tilde{W}_\infty(t)\sigma \in \bar{N} \cap \sigma^{-1}\bar{N}\sigma$ and $\sigma^{-1} n(t)\sigma W_0(t) \in B$. We rewrite the decomposition (2.3) as

$$W_\infty(t)^{-1} W_0(t) = \sigma^{-1} \mathrm{e}^{tL}, \tag{2.4}$$

where $W_\infty(t) \in \sigma^{-1}\bar{N}\sigma \cap \bar{N}$ and $W_0(t) \in B$. Put $L(t) = W_\infty(t) L W_\infty(t)^{-1}$. We obtain

$$\dot{L}(t) = [-(W_\infty(t)\sigma^{-1} L \sigma W_\infty(t)^{-1})_-, L(t)] \tag{2.5}$$

from (2.4). Since the right hand side of (1.5) leaves the σ-cell for the ϕ-cell at $t = 0$, $W_\infty(t)$ of (2.4) has a pole at $t = 0$ [9].

3. The Toda lattice as the Hamiltonian system on the coordinate ring of Hessenberg matrices

Let V be the affine space of the $n \times n$ Hessenberg matrices

$$V = \left\{ \Lambda + \sum_{1 \le j \le i \le n} L_{i,j} E_{i,j} | L_{i,j} \in \mathbf{C} \right\}.$$

Let $\mathcal{O}$ be a sheaf of the commutative algebra on V and $\mathcal{I}$ be a sheaf of the ideal of $\mathcal{O}$. The sheaf of the quotient algebra $\mathcal{O}/\mathcal{I}$ is the sheaf associated with the presheaf whose section on U is $\Gamma(U, \mathcal{O})/\Gamma(U, \mathcal{I})$ for any open set U of V. Let C^ω be the sheaf of the analytic functions on V. We define the subsheaf $\mathcal{C}$ of C^ω on V as follows. Let U be an open set of V. The section $\Gamma(U, \mathcal{C})$ is the commutative algebra generated by the coordinate functions $\mathcal{L}_{i,j}(U)$, $1 \le j \le i \le n$, where $\mathcal{L}_{i,j}(U)(L) = L_{i,j}$ for $L \in U$. As long as we do not need manifestation, we abbreviate $\mathcal{L}_{i,j}(U)$ as $\mathcal{L}_{i,j}$ and U is an arbitrary open set of V. We now define the sheaf of the formal power series $\mathcal{O}[[t]]$ whose section on U is $\Gamma(U, \mathcal{O}[[t]]) = \Gamma(U, \mathcal{O})[[t]]$. We define the subsheaf of $\Lambda + \mathcal{C} \otimes \bar{\mathfrak{b}}$, *Lax*, such that

$$\Gamma(U, Lax) = \left\{ \Lambda + \sum_{1 \le j \le i \le n} \mathcal{L}_{i,j} \otimes E_{i,j} \right\}. \tag{3.1}$$

We have the following decomposition of $\mathcal{C}$:

$$\Gamma(U, \mathcal{C}) = \Gamma(U, \mathcal{A}) \otimes \Gamma(U, \mathcal{C}^{\bar{N}}), \tag{3.2}$$

where $\mathcal{A}$ is the sheaf of the subalgebra of $\mathcal{C}$ which is isomorphic to the sheaf of the affine algebra of $\bar{N}$ and $\mathcal{C}^{\bar{N}}$ is the sheaf of the subalgebra of $\mathcal{C}$ of invariants with respect to the adjoint action of $\bar{N}$. The sheaf $\mathcal{C}^{\bar{N}}$ is the sheaf associated with the presheaf $U \to \Gamma(U, \mathcal{C})^{\bar{N}}$ for any open set U of V. This decomposition is valid for any complex semi-simple Lie algebra [12]. There exist $\mathcal{W}_\infty \in \Gamma(U, \mathcal{A}) \otimes \bar{N}$ and $\chi_0 \in \Gamma(U, \mathcal{C}^{\bar{N}}) \otimes \mathrm{Mat}(n, \mathbf{C})$ uniquely given in a form such as

$$\chi_0 = \Lambda + \sum_{i=1}^{n} \varphi_i \otimes E_{i,1}, \quad \varphi_1, \ldots, \varphi_n \in \Gamma(U, \mathcal{C}^{\bar{N}})$$

and $\mathcal{L} = \mathcal{W}_\infty \chi_0 \mathcal{W}_\infty^{-1}$ for any $\mathcal{L} \in \Gamma(U, Lax)$. There exists a structure of the Poisson algebra of Kostant and Kirillov in $\Gamma(U, \mathcal{C})$ defined by

$$\{\mathcal{L}_{i,j}, \mathcal{L}_{k,\ell}\} = \delta_{j,k}\mathcal{L}_{i,\ell} - \delta_{\ell,i}\mathcal{L}_{k,j}. \tag{3.3}$$

We define the vector field $\mathcal{X}$ on $\Gamma(U, \mathcal{C})$ by

$$\mathcal{X} f = \left\{ \frac{1}{2}\mathrm{tr}\mathcal{L}^2, f \right\}, \quad \text{for } f \in \Gamma(U, \mathcal{C}). \tag{3.4}$$

We see that

$$\mathcal{X}\mathcal{L} = [\mathcal{L}_+, \mathcal{L}]. \tag{3.5}$$

Let us construct the sheaf of solutions of the full Kostant–Toda lattice as a subsheaf of $\Lambda + \mathcal{C}[[t]] \otimes \bar{\mathfrak{b}}$. For $\mathcal{L}(t) \in \Lambda + \Gamma(U, \mathcal{C})[[t]] \otimes \bar{\mathfrak{b}}$, we define the tangent vector $\mathcal{X}(t) \in T_{\mathcal{L}(t)}\Gamma(U, \mathcal{C})[[t]]$ by

$$\mathcal{X}(t) = \left\{\frac{1}{2}\mathrm{tr}\mathcal{L}(t)^2, *\right\}.$$

$\mathcal{X}(t)$ transforms the sections of $\Gamma(U, \mathcal{C})[[t]]$ to the sections of $\Gamma(U, \mathcal{C})[[t]]$.

Lemma 3.2. *There exists a unique solution of the Hamiltonian equation*

$$\frac{\mathrm{d}\mathcal{L}(t)}{\mathrm{d}t} = \mathcal{X}(t)\mathcal{L}(t) \tag{3.6}$$

as a section of $\Lambda + \Gamma(U, \mathcal{C})[[t]] \otimes \bar{\mathfrak{b}}$.

Proof. Put

$$\mathcal{L}(t) = \mathcal{L} + t\mathcal{B}^{(1)} + t^2\mathcal{B}^{(2)} + \cdots,$$

where $\mathcal{B}^{(i)} \in \Gamma(U, \mathcal{C}) \otimes \mathrm{Mat}(n, \mathbf{C})$. Then we can determine $\mathcal{B}^{(i)} \in \Gamma(U, \mathcal{C}) \otimes \bar{b}$, $i = 1, 2, \ldots$, uniquely. □

We define the sheaf Sol by

$$\Gamma(U, \mathrm{Sol}) = \{\mathcal{L}(t) | \mathcal{L}(t) \in \Lambda + \Gamma(U, \mathcal{C})[[t]] \otimes \bar{\mathfrak{b}} \text{ satisfies (3.6)}\}.$$

Let $\mathcal{L}(t) = (\mathcal{L}_{i,j}(t))_{1 \le j \le i \le n}$ be the section of $\Gamma(U, \mathrm{Sol})$. We will show that the Hamiltonian flow of $\mathcal{X}(t)$ preserves the Poisson bracket.

Lemma 3.3. *It holds that*

$$\{\mathcal{L}_{i,j}(t), \mathcal{L}_{k,\ell}(t)\} = \delta_{j,k}\mathcal{L}_{i,\ell}(t) - \delta_{\ell,i}\mathcal{L}_{k,j}(t). \tag{3.7}$$

Proof. In the same way as in the proof of Lemma 3.2, there exists a unique solution of

$$\frac{\mathrm{d}f(t)}{\mathrm{d}t} = \mathcal{X}(t)f(t)$$

in $\Gamma(U, \mathcal{C})[[t]]$. Put $F(t) = \{\mathcal{L}_{i,j}(t), \mathcal{L}_{k,\ell}(t)\}$ and $G(t) = \delta_{j,k}\mathcal{L}_{i,\ell}(t) - \delta_{\ell,i}\mathcal{L}_{k,j}(t)$. We have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}F(t) &= \left\{\frac{\mathrm{d}\mathcal{L}_{i,j}(t)}{\mathrm{d}t}, \mathcal{L}_{k,\ell}(t)\right\} + \left\{\mathcal{L}_{i,j}(t), \frac{\mathrm{d}\mathcal{L}_{k,\ell}(t)}{\mathrm{d}t}\right\} = \{\mathcal{X}(t)\mathcal{L}_{i,j}(t), \mathcal{L}_{k,\ell}(t)\} + \{\mathcal{L}_{i,j}(t), \mathcal{X}(t)\mathcal{L}_{k,\ell}(t)\} \\ &= \mathcal{X}(t)\{\mathcal{L}_{i,j}(t), \mathcal{L}_{k,\ell}(t)\} = \mathcal{X}(t)F(t).\end{aligned}$$

On the other hand it is clear that

$$\frac{\mathrm{d}}{\mathrm{d}t}G(t) = \mathcal{X}(t)G(t).$$

Since $F(0) = G(0)$, we have $F(t) = G(t)$ by the uniqueness of the solution. □

We have the following proposition from (3.5) and Lemma 3.3.

Proposition 3.4. *The Hamiltonian equation* (3.6) *induces the Lax equation*

$$\frac{\mathrm{d}\mathcal{L}(t)}{\mathrm{d}t} = [\mathcal{L}(t)_+, \mathcal{L}(t)]. \tag{3.8}$$

Let us show the fact that the Hamiltonian vector field $\mathcal{X}(t)$ is tangential to the submanifold of Jacobi elements in sheaf theoretical language. This is a preparation for the proof of the main theorem. Let $\mathcal{I}$ be the sheaf of the ideal of $\mathcal{C}$ where $\Gamma(U, \mathcal{I})$ is generated by $\mathcal{L}_{i,j}$ with $i - j \ge 2$. By definition of the Poisson bracket, it is easy to see that $\mathcal{I}$ is also an ideal with respect to the Poisson bracket, that is,

$$\{\mathcal{C}, \mathcal{I}\} \subset \mathcal{I}.$$

From this fact there exists a structure of Poisson algebra in the sheaf of the quotient algebra $\mathcal{C}/\mathcal{I}$. Let $\rho : \mathcal{C} \to \mathcal{C}/\mathcal{I}$ be the canonical morphism of sheaves. Note that ρ is also a morphism with respect to the Poisson bracket.

Remark. Let $\bar{N}_3$ be the subgroup of $\bar{N}$ defined by

$$\bar{N}_3 := \{W = (w_{i,j}) \in \bar{N} | w_{i,j} = 0 \text{ for } 1 \le i - j \le 2\}.$$

There exists a Hamiltonian action of $\bar{N}_3$ on $V(\mathrm{Ad}\bar{N}_3)$. Let us consider the Hamiltonian reduction [14] $V/\bar{N}_3$. Let $\pi : V \to V/\bar{N}_3$ be a canonical projection. We can define the sheaf of the affine coordinate ring of $V/\bar{N}_3$. We denote it by $\mathcal{C}'$. There exists an isomorphism of sheaves on V between $\mathcal{C}/\mathcal{I}$ and $\pi^{-1}\mathcal{C}'$.

For $\mathcal{P} = (\mathcal{P}_{i,j}) \in \Gamma(U, \mathcal{C}) \otimes \mathrm{Mat}(n, \mathbf{C})$, we use the notation $\rho(\mathcal{P}) = (\rho(\mathcal{P}_{i,j}))$. Strictly speaking, we should use the notation $\rho(U)$. However we abbreviate $\rho(U)$ as ρ as long as there is no confusion. Since the sections of $\Gamma(U, \mathrm{Sol})$ are expanded with respect to t into a form such as

$$\mathcal{L}(t) = \mathcal{L}(0) + t\mathcal{B}^{(1)} + t^2\mathcal{B}^{(2)} + \cdots,$$

where $\mathcal{L}(0) = \mathcal{L} \in \Gamma(U, Lax)$ and $\mathcal{B}^{(j)} \in \Gamma(U, \mathcal{C}) \otimes \bar{\mathfrak{b}}$, $j = 1, 2, \ldots$, we can extend ρ as

$$\rho(\mathcal{L}(t)) = \rho(\mathcal{L}) + t\rho(\mathcal{B}^{(1)}) + t^2\rho(\mathcal{B}^{(2)}) + \cdots.$$

Proposition 3.5. *Suppose* $\mathcal{L}(t) = (\mathcal{L}_{i,j}(t)) \in \Gamma(U, \mathrm{Sol})$. *It holds that*

$$\rho(\mathcal{L}_{i,j}(t)) = 0, \tag{3.9}$$

for $i - j \ge 2$.

Proof. Put

$$\mathcal{L}(t) = \mathcal{L} + t\mathcal{B}^{(1)} + t^2\mathcal{B}^{(2)} + \cdots.$$

Thus we see that

$$\mathcal{B}^{(k)} = \frac{1}{k!}\left.\frac{\mathrm{d}^k\mathcal{L}(t)}{\mathrm{d}t^k}\right|_{t=0}. \tag{3.10}$$

We show the following lemma.

Lemma 3.6. *The derivatives* $\frac{\mathrm{d}^k\mathcal{L}(t)}{\mathrm{d}t^k}$, $k = 1, 2, \ldots$, *are linear sums of terms such as*

$$[\Pi_1(t)_+, [\Pi_2(t)_+, [\ldots, [\Pi_\mu(t)_+, \mathcal{L}(t)]\ldots]]], \tag{3.11}$$

where $1 \le \mu \le k$ *and* $\Pi_i(t) \in \Gamma(U, \mathcal{C})[[t]] \otimes \mathrm{Mat}(n, \mathbf{C})$.

Proof. We show this lemma by induction on k. When k is 1, this lemma is straightforward from (3.8). Suppose $\frac{\mathrm{d}^k\mathcal{L}(t)}{\mathrm{d}t^k}$ is a linear sum of terms of (3.11). Thus $\frac{\mathrm{d}^{k+1}\mathcal{L}(t)}{\mathrm{d}t^{k+1}}$ is a linear sum of terms:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}[\Pi_1(t)_+, [\Pi_2(t)_+, [\ldots, [\Pi_\mu(t)_+, \mathcal{L}(t)]\ldots]]] &= \left[\left(\frac{\mathrm{d}\Pi_1(t)}{\mathrm{d}t}\right)_+, [\Pi_2(t)_+, [\ldots, [\Pi_\mu(t)_+, \mathcal{L}(t)]\ldots]]\right]\\ &+ \cdots + \left[\Pi_1(t)_+, \left[\Pi_2(t)_+, \left[\ldots, \left[\left(\frac{\mathrm{d}\Pi_\mu(t)}{\mathrm{d}t}\right)_+, \mathcal{L}(t)\right]\ldots\right]\right]\right]\\ &+ [\Pi_1(t)_+, [\Pi_2(t)_+, [\ldots, [\Pi_\mu(t)_+, [\mathcal{L}(t)_+, \mathcal{L}(t)]]\ldots]]].\end{aligned}$$

This completes the proof of Lemma 3.6. □

We see that $\mathcal{B}^{(k)}$ is a linear combination of terms such as

$$[\Pi_1(0)_+, [\Pi_2(0)_+, [\ldots, [\Pi_\mu(0)_+, \mathcal{L}]\ldots]]] \tag{3.12}$$

from Lemma 3.6 and (3.10). We will show that the ij components of $\mathcal{B}^{(k)}$ belong to $\Gamma(U,\mathcal{I})\otimes\bar{\mathfrak{b}}$ if $i-j\geq 2$ by induction. Let Π_1 be a matrix of $\Gamma(U,\mathcal{C})\otimes\mathrm{Mat}(n,\mathbf{C})$. Thus the ij component of $[\Pi_{1+},\mathcal{L}]$ is the ij component of

$$\left[\Pi_{1+},\sum_{k-\ell\geq 2}\mathcal{L}_{k,\ell}E_{k,\ell}\right].$$

Then we have $([\Pi_{1+},\mathcal{L}])_{i,j}\in\Gamma(U,\mathcal{I})$ if $i-j\geq 2$. Suppose

$$([\Pi_{1+},[\Pi_{2+},[\ldots,[\Pi_{\mu+},\mathcal{L}]\ldots]]])_{i,j}\in\Gamma(U,\mathcal{I}),$$

for $i-j\geq 2$. We see that

$$\begin{aligned}&([\Pi_{1+},[\Pi_{2+},[\ldots,[\Pi_{\mu+},[\Pi_{\mu+1_+},\mathcal{L}]]\ldots]]])_{i,j}\\&\quad=\left(\left[\Pi_{1+},\sum_{k-\ell\geq i-j}([\Pi_{2+},[\ldots,[\Pi_{\mu+1_+},\mathcal{L}]\ldots]])_{k,\ell}E_{k,\ell}\right]\right)_{i,j}.\end{aligned}\tag{3.13}$$

Since $i-j\geq 2$, we have $k-\ell\geq 2$. We see that

$$([\Pi_{2+},[\ldots,[\Pi_{\mu+},\mathcal{L}]\ldots]])_{k,\ell}\in\Gamma(U,\mathcal{I})$$

from the assumption of the induction. Then we see that the left hand side of (3.13) is an element of $\Gamma(U,\mathcal{I})$. From this fact we see that

$$\mathcal{L}_{i,j}(t)=\mathcal{L}_{i,j}+\sum_{k=1}^{\infty}t^k\mathcal{B}^{(k)}_{i,j}$$

belongs to $\Gamma(U,\mathcal{I})[[t]]$ if $i-j\geq 2$. Then we have $\rho(\mathcal{L}_{i,j}(t))=0$ for $i-j\geq 2$. □

Proposition 3.7. *Let $\mathcal{L}(t)$ be a section of the solution of $\Gamma(U,\mathrm{Sol})$. Then $\rho(\mathcal{L}(t))$ satisfies the Lax equation*

$$\frac{\mathrm{d}\rho(\mathcal{L}(t))}{\mathrm{d}t}=[\rho(\mathcal{L}(t))_+,\rho(\mathcal{L}(t))].\tag{3.14}$$

Furthermore (3.14) is the Hamiltonian equation for $\frac{1}{2}\mathrm{tr}\rho(\mathcal{L}(t))^2$ on $\Gamma(U,\mathcal{C}/\mathcal{I})[[t]]$.

Proof. By the definition of ρ, we see that

$$\frac{\mathrm{d}\rho(\mathcal{L}(t))}{\mathrm{d}t}=\rho\left(\frac{\mathrm{d}\mathcal{L}(t)}{\mathrm{d}t}\right).$$

Since ρ is an algebraic homomorphism, we have

$$\frac{\mathrm{d}\rho(\mathcal{L}(t))}{\mathrm{d}t}=\rho\left(\frac{\mathrm{d}\mathcal{L}(t)}{\mathrm{d}t}\right)=\rho([\mathcal{L}(t)_+,\mathcal{L}(t)])=[\rho(\mathcal{L}(t))_+,\rho(\mathcal{L}(t))].$$

On the other hand since ρ is also a morphism of a sheaf of the Poisson algebra, we have

$$\frac{\mathrm{d}\rho(\mathcal{L}(t))}{\mathrm{d}t}=\rho\left(\frac{\mathrm{d}\mathcal{L}(t)}{\mathrm{d}t}\right)=\rho\left(\left\{\frac{1}{2}\mathrm{tr}\mathcal{L}^2(t),\mathcal{L}(t)\right\}\right)=\left\{\frac{1}{2}\mathrm{tr}\rho(\mathcal{L}(t))^2,\rho(\mathcal{L}(t))\right\}.\quad\square$$

Note that $\rho(\mathcal{L}(t))$ is a Jacobi element from Proposition 3.5 Then (3.14) is the Lax equation of the Toda lattice (not the full Kostant–Toda lattice).

4. Wave operators and τ functions

In this section we consider the Gauss decomposition (1.7)

$$\tilde{W}_\infty(t)^{-1}\tilde{W}_0(t)=\mathrm{e}^{tL}\sigma^{-1},\tag{4.1}$$

where $\tilde{W}_\infty(t) \in \bar{N}$ and $\tilde{W}_0(t) \in B \cap \sigma B \sigma^{-1}$. Let us show that the decomposition (4.1) is equivalent to the decomposition $W_\infty(t)^{-1}\sigma W_0(t) = e^{tL}$. In the same way as in Section 2, there exists $b(t) \in N$ such that $\sigma b(t)\sigma^{-1} \in \bar{N}$ and $\sigma b(t) W_0(t)\sigma^{-1} \in B$. Put $\tilde{W}_0(t) = b(t)W_0(t)$. Since $W_\infty(t)^{-1}\sigma W_0(t) = e^{tL}$, we have

$$(W_\infty(t)^{-1}\sigma b(t)^{-1}\sigma^{-1})(\sigma \tilde{W}_0(t)\sigma^{-1}) = e^{tL}\sigma^{-1}.$$

Since $\sigma b(t)\sigma^{-1} \in \bar{N}$ and $W_\infty(t) \in \bar{N}$, we have $\tilde{W}_\infty(t) \in \bar{N}$, where $\tilde{W}_\infty(t) = \sigma b(t)\sigma^{-1}W_\infty(t)$. On the other hand, since $\tilde{W}_0(t) \in B$ and $\sigma \tilde{W}_0(t)\sigma^{-1} \in B$, we have $\sigma \tilde{W}_0(t)\sigma^{-1} \in B \cap \sigma B\sigma^{-1}$. Let us show the converse. Suppose that we have the decomposition (4.1). We have

$$\tilde{W}_\infty(t)\sigma(\sigma^{-1}\tilde{W}_0(t)\sigma) = e^{tL}.$$

Since $\tilde{W}_0(t) \in B \cap \sigma B \sigma^{-1}$, we recover the decomposition of the σ-cell $\bar{N}\sigma B/B$. This decomposition relates to a cell decomposition of $\bar{N}B$ such as $\bar{N}B = \sqcup_{\sigma\in\mathcal{S}_n} Y'_\sigma$, where Y'_σ is an open dense subset of $\pi'_\sigma = \bar{N}(B \cap \sigma B\sigma^{-1})$ defined by $Y'_\sigma = \pi'_\sigma - \cup_{\tau>\sigma}\pi'_\tau$. Put $\tilde{L}(t) = \tilde{W}_\infty(t)L\tilde{W}_\infty(t)^{-1}$. In this case we obtain the ordinary Lax equation $\dot{\tilde{L}}(t) = [\tilde{L}(t)_+, \tilde{L}(t)]$ from (4.1) in comparison with (2.5). Let us consider the problem mentioned in the introduction. As we said in the introduction, $\tilde{L}(t)$ is a Jacobi element if L is a Jacobi element in the case of $\sigma = id$ [13]. The key point of this proof is $e^{tL} \in G^L$, a centralizer of L. However in the case of $\sigma \neq id$, $e^{tL}\sigma^{-1} \notin G^L$ in general. Let U be an open subset of V. The degree θ in $\Gamma(U, \mathcal{C})$ is defined by

$$\theta(\mathcal{L}^{m_1}_{i_1,j_1}\cdots\mathcal{L}^{m_r}_{i_r,j_r}) = \sum_{\mu=1}^{r} m_\mu(i_\mu - j_\mu + 1).$$

We call this grading the x_0-grading of Kazhdan ([12], p. 111). Let $\mathcal{C}_\mu$ be the sheaf of a homogeneous space of degree μ such that $\Gamma(U, \mathcal{C}_\mu) = \{f \in \Gamma(U,\mathcal{C}) | \theta(f) = \mu\}$. We regard $\mathcal{C}_0 = \mathbf{C}$ as a constant sheaf on V. We define the completion $\bar{\mathcal{C}}$ of $\mathcal{C}$ by

$$\Gamma(U, \bar{\mathcal{C}}) = \left\{\sum_{\mu=0}^{\infty} a_\mu f_\mu | a_\mu \in \mathbf{C}, f_\mu \in \Gamma(U, \mathcal{C}_\mu)\right\}.$$

For $X = \sum_\mu a_\mu f_\mu \in \Gamma(U, \bar{\mathcal{C}})$ and $Y = \sum_\mu b_\mu g_\mu \in \Gamma(U, \bar{\mathcal{C}})$, where $a_\mu, b_\mu \in \mathbf{C}$ and $f_\mu, g_\mu \in \Gamma(U, \mathcal{C}_\mu)$, the product $XY \in \Gamma(U, \bar{\mathcal{C}})$ is defined by

$$XY = \sum_{\zeta=0}^{\infty}\sum_{\mu+\nu=\zeta} a_\mu b_\nu f_\mu g_\nu.$$

Then $\bar{\mathcal{C}}$ becomes the sheaf of the commutative algebra on V. We define $\exp\mathcal{L}$, a subsheaf of $\bar{\mathcal{C}}$, as follows.

Lemma 4.1. *Put $\Phi(\mathcal{L}) = e^{\mathcal{L}} = (\Phi_{i,j}(\mathcal{L}))_{1\le i,j\le n}$ for $\mathcal{L} \in \Gamma(U, Lax)$. Then $\Phi_{i,j}(\mathcal{L})$ is a section of $\Gamma(U, \bar{\mathcal{C}})$.*

Proof. $\Phi(\mathcal{L})$ is formally expressed by $\Phi(\mathcal{L}) = \sum_{\mu=0}^{\infty}\mathcal{L}^\mu/\mu!$. Put $\mathcal{L}^\mu = (\mathcal{L}^{(\mu)}_{i,j})_{i,j}$. Then we see that

$$\theta(\mathcal{L}^{(\mu)}_{i,j}) = i - j + \mu. \qquad (*)$$

We show this fact by induction. When $\mu = 1$, we have

$$\mathcal{L}^{(1)}_{i,j} = \begin{cases} \mathcal{L}_{i,j} & i \ge j \\ 1 & j = i+1 \\ 0 & j > i+1. \end{cases}$$

Then we have $\theta(\mathcal{L}^{(1)}_{i,j}) = i - j + 1$, where we take $a = 0$ if $\theta(a) < 0$. We assume $(*)$ is correct at μ. Since

$$\mathcal{L}^{\mu+1} = \begin{pmatrix} \mathcal{L}_{1,1} & 1 & 0\ldots & \\ \vdots & \ddots & \ddots & \vdots \\ \mathcal{L}_{n,1} & \ldots & \ldots & \mathcal{L}_{n,n} \end{pmatrix} \begin{pmatrix} \mathcal{L}^{(\mu)}_{1,1} & \ldots & \mathcal{L}^{(\mu)}_{1,n} \\ \vdots & \ldots & \vdots \\ \mathcal{L}^{(\mu)}_{n,1} & \ldots & \mathcal{L}^{(\mu)}_{n,n} \end{pmatrix},$$

we have

$$(\mathcal{L}^{\mu+1})_{i,j} = \sum_{k=1}^{i} \mathcal{L}_{i,k}\mathcal{L}^{(\mu)}_{k,j} + \mathcal{L}^{(\mu)}_{i+1,j}.$$

We have

$$\theta(\mathcal{L}_{i,k}\mathcal{L}^{(\mu)}_{k,j}) = i - k + 1 + k - j + \mu = i - j + (\mu + 1)$$

and $\theta(\mathcal{L}^{(\mu)}_{i+1,j}) = i + 1 - j + \mu = i - j + (\mu + 1)$ from the assumption of the induction. Then we show that $(*)$ is correct. Since $\Phi_{i,j}(\mathcal{L}) = \sum_{\mu=0}^{\infty} \frac{1}{\mu!}\mathcal{L}^{(\mu)}_{i,j}$ and $(*)$, we have $\Phi_{i,j}(\mathcal{L}) \in \Gamma(U, \bar{\mathcal{C}})$. □

Definition. For any fixed finite value $t_0 \in \mathbf{C}$, $\exp(t_0\mathcal{L})$ is a subsheaf of $\bar{\mathcal{C}}$ on V such as

$$\Gamma(U, \exp(t_0\mathcal{L})) = \text{ algebra generated by } \Phi_{i,j}(t_0\mathcal{L}), \quad 1 \le i, j \le n.$$

Remark. We fix $t_0 \in \mathbf{C}$ in the Definition. Then $\Phi_{i,j}(t_0\mathcal{L})$ is not a section of $\Gamma(U, \mathcal{C})[[t]]$ but a section of $\Gamma(U, \bar{\mathcal{C}})$. The proof of the fact that $\Phi_{i,j}(t_0\mathcal{L}) \in \Gamma(U, \bar{\mathcal{C}})$ is the same as the proof of Lemma 4.1.

Let $\mathcal{O}$ be a sheaf of the commutative algebra on V. Let $\mathcal{Q}(\mathcal{O})$ be the sheaf of the quotient field of $\mathcal{O}$. We denote $\mathcal{Q}(\mathcal{O}[[t]])$ by $\mathcal{O}((t))$. Put $\Phi(t) = (\Phi_{i,j}(t))_{i,j} = \mathrm{e}^{t\mathcal{L}}$. Then we see that $\Phi_{i,j}(t) \in \Gamma(U, \mathcal{C})[[t]]$ and $\Phi_{i,j}(t_0) \in \Gamma(U, \bar{\mathcal{C}})$ for any finite value t_0. Let C^{ω} be the sheaf of the analytic function on V. Since e^{tL} converges for any t, $\exp(t_0\mathcal{L})$ is the sheaf of the subalgebra of C^{ω}. Let $\mathcal{F}$ be a certain sheaf on V. Let us consider the Gauss decomposition

$$\mathcal{W}_{\infty}(t)^{-1}\mathcal{W}_0(t) = \mathrm{e}^{(t-t_0)\mathcal{L}}\sigma^{-1}, \tag{4.2}$$

where $\mathcal{W}_{\infty}(t) \in \Gamma(U, \mathcal{F}) \otimes \bar{N}$ and $\mathcal{W}_0(t) \in \Gamma(U, \mathcal{F}) \otimes B \cap \sigma B\sigma^{-1}$. We will determine $\mathcal{F}$. Note that $\mathcal{W}_{\infty}(t)$ has a pole at $t = t_0$ in (4.2). To consider $\mathcal{W}_{\infty}(0)$, we move the pole from $t = 0$ to $t = t_0$. We define the *weight* on $\Gamma(U, \mathcal{C})[[t - t_0]]$ as follows. Suppose $(t - t_0)^n g \in \Gamma(U, \mathcal{C})[[t - t_0]]$ and g is a homogeneous element for θ. Then we define $weight((t - t_0)^n g) = \theta(g) - n$. Put $\Phi(t) = \mathrm{e}^{(t-t_0)\mathcal{L}} = (\Phi_{i,j}(t))_{i,j}$. We have the following lemma.

Lemma 4.2. *It holds that*

$$weight(\Phi_{i,j}(t)) = i - j.$$

Proof. Since $\Phi_{i,j}(t) = \sum_{\mu=0}^{\infty} \frac{(t-t_0)^{\mu}}{\mu!}\mathcal{L}^{(\mu)}_{i,j}$ and $weight((t - t_0)^{\mu}\mathcal{L}^{(\mu)}_{i,j}) = i - j + \mu - \mu = i - j$, we have the conclusion. □

We have the following lemma.

Lemma 4.3. *Suppose* $f(t) = \sum_{\mu=0}^{\infty} f_{\mu}t^{\mu} \in \Gamma(U, \mathcal{C})[[t]]$ *satisfies a condition such that* $weight(f_{\mu}t^{\mu})$ *is constant on* μ. *Then* $f(t_0)$ *is an element of* $\Gamma(U, \bar{\mathcal{C}})$ *for any finite value* t_0.

Proof. Let M be the constant of this proposition. Since $\theta(f_{\mu}) = M + \mu$, we have $f_{\mu}t_0^{\mu} \in \mathcal{C}_{M+\mu}$. This means $f(t_0) \in \bar{\mathcal{C}}$ for any finite value t_0. □

Put $\Psi(t) = (\psi_{i,j}(t))_{i,j} = \mathrm{e}^{(t-t_0)\mathcal{L}}\sigma^{-1}$.

Lemma 4.4. *It holds that*

$$weight\left(\begin{vmatrix} \psi_{1,1}(t) & \dots & \psi_{1,j-1}(t) \\ \vdots & \dots & \vdots \\ \psi_{j-1,1}(t) & \dots & \psi_{j-1,j-1}(t) \end{vmatrix}\right) = \frac{j(j-1)}{2} - \{\sigma^{-1}(1) + \dots + \sigma^{-1}(j-1)\}.$$

Proof. We see that

$$\begin{vmatrix} \psi_{1,1}(t) & \dots & \psi_{1,j-1}(t) \\ \vdots & \dots & \vdots \\ \psi_{j-1,1}(t) & \dots & \psi_{j-1,j-1}(t) \end{vmatrix} = \begin{vmatrix} \Phi_{1,\sigma^{-1}(1)}(t) & \dots & \Phi_{1,\sigma^{-1}(j-1)}(t) \\ \vdots & \dots & \vdots \\ \Phi_{j-1,\sigma^{-1}(1)} & \dots & \Phi_{j-1,\sigma^{-1}(j-1)}(t) \end{vmatrix}$$
$$= \sum_{\tau \in \mathcal{S}_{j-1}} \operatorname{sgn} \tau \, \Phi_{1,\tau\sigma^{-1}(1)}(t) \cdots \Phi_{j-1,\tau\sigma^{-1}(j-1)}(t),$$

where we regard $\tau \in \mathcal{S}_{j-1}$ as a permutation of $\{\sigma^{-1}(1), \dots, \sigma^{-1}(j-1)\}$. From Lemma 4.2 it holds that $weight(\Phi_{i,\tau\sigma^{-1}(i)}(t)) = i - \tau\sigma^{-1}(i)$; then we have

$$weight(\Phi_{1,\tau\sigma^{-1}(1)}(t) \cdots \Phi_{j-1,\tau\sigma^{-1}(j-1)}(t)) = \frac{j(j-1)}{2} - \{\tau\sigma^{-1}(1) + \dots + \tau\sigma^{-1}(j-1)\}$$
$$= \frac{j(j-1)}{2} - \{\sigma^{-1}(1) + \dots + \sigma^{-1}(j-1)\}. \quad \square$$

We have

$$(\mathcal{W}_\infty(t) \mathrm{e}^{(t-t_0)\mathcal{L}} \sigma^{-1})_- = 0, \tag{4.3}$$

from (4.2). Put $\mathcal{W}_\infty(t) = (w^\infty_{i,j}(t))_{i,j}$. Then we have

$$(w^\infty_{j,1}(t), \dots, w^\infty_{j,j-1}(t)) \begin{pmatrix} \psi_{1,1}(t) & \dots & \psi_{1,j-1}(t) \\ \vdots & \dots & \vdots \\ \psi_{j-1,1}(t) & \dots & \psi_{j-1,j-1}(t) \end{pmatrix} = -(\psi_{j,1}(t), \dots, \psi_{j,j-1}(t)). \tag{4.4}$$

We have $w^\infty_{j,k}(t) = -\tau_{j,k}(t)/\tau_j(t)$ from (4.4), where

$$\tau_j(t) = \begin{vmatrix} \psi_{1,1}(t) & \dots & \psi_{1,j-1}(t) \\ \vdots & \dots & \vdots \\ \psi_{j-1,1}(t) & \dots & \psi_{j-1,j-1}(t) \end{vmatrix} \quad \text{and} \quad \tau_{j,k}(t) = \begin{vmatrix} \psi_{1,1}(t) & \dots & \psi_{1,j-1}(t) \\ \vdots & \dots & \vdots \\ \psi_{k-1,1}(t) & \dots & \psi_{k-1,j-1}(t) \\ \psi_{j,1}(t) & \dots & \psi_{j,j-1}(t) \\ \psi_{k+1,1}(t) & \dots & \psi_{k+1,j-1}(t) \\ \vdots & \dots & \vdots \\ \psi_{j-1,1}(t) & \dots & \psi_{j-1,j-1}(t) \end{vmatrix}.$$

Then we see that $\mathcal{W}_\infty(t) \in \Gamma(U, \mathcal{C})((t-t_0)) \otimes \bar{N}$. We see that $\mathcal{W}_\infty(0) \in \Gamma(U, \mathcal{Q}(\bar{\mathcal{C}})) \otimes \bar{N}$ from Lemmas 4.3 and 4.4. Moreover we see that $\mathcal{W}_\infty(0) \in \Gamma(U, \mathcal{Q}(\exp(-t_0\mathcal{L}))) \otimes \bar{N} \subset \Gamma(U, \mathcal{Q}(C^\omega)) \otimes \bar{N}$. On the other hand from

$$\mathcal{W}_0(t) = \mathcal{W}_\infty(t) \mathrm{e}^{(t-t_0)\mathcal{L}} \sigma^{-1},$$

we have

$$\mathcal{W}_0(t) \in \Gamma(U, \mathcal{C})((t-t_0)) \otimes (B \cap \sigma B \sigma^{-1})$$

and

$$\mathcal{W}_0(0) \in \Gamma(U, \mathcal{Q}(\exp(-t_0\mathcal{L}))) \otimes (B \cap \sigma B \sigma^{-1}) \subset \Gamma(U, \mathcal{Q}(C^\omega)) \otimes (B \cap \sigma B \sigma^{-1}).$$

$\mathcal{L}$ can be decomposed into a form such as $\mathcal{L} = \mathcal{W}_\infty \chi_0 \mathcal{W}_\infty^{-1}$, where $\mathcal{W}_\infty \in \Gamma(U, \mathcal{C}) \otimes \bar{N}$ and the components of the first column of χ_0 are generators of $\Gamma(U, \mathcal{C}^{\bar{N}})$. Note that $\mathcal{W}_\infty \neq \mathcal{W}_\infty(0)$. We have

$$\tilde{\mathcal{W}}_\infty(t)^{-1} \mathcal{W}_0(t) = \mathrm{e}^{(t-t_0)\chi_0} \mathcal{W}_\infty^{-1} \sigma^{-1}, \tag{4.5}$$

from (4.2), where $\tilde{\mathcal{W}}_\infty(t) = \mathcal{W}_\infty(t)\mathcal{W}_\infty$. Put $\tilde{\mathcal{L}}(t) = \tilde{\mathcal{W}}_\infty(t) \chi_0 \tilde{\mathcal{W}}_\infty^{-1}(t)$. We see that $\tilde{\mathcal{L}}(t)$ has the form $\Lambda + \bar{\mathfrak{b}}$. Put $\tilde{\mathcal{L}} = (\tilde{\mathcal{L}}_{i,j})_{i,j} = \tilde{\mathcal{L}}(0)$. Let $\tilde{\mathcal{C}}$ be the sheaf on V whose section on U is the algebra generated by $\tilde{\mathcal{L}}_{i,j}$, $1 \leq j \leq i \leq n$.

Since $\tilde{\mathcal{L}} = \tilde{\mathcal{W}}_\infty(0)\chi_0\tilde{\mathcal{W}}_\infty(0)^{-1}$, we see that $\varphi_1, \ldots, \varphi_n \in \Gamma(U, \tilde{\mathcal{C}}^{\bar{N}})$ and $\tilde{\mathcal{W}}_\infty(0) \in \Gamma(U, \tilde{\mathcal{C}}) \otimes \bar{N}$. From (4.5), we see that $\tilde{\mathcal{L}}(t)$ satisfies the Lax equation

$$\dot{\tilde{\mathcal{L}}}(t) = [\tilde{\mathcal{L}}(t)_+, \tilde{\mathcal{L}}(t)]. \tag{4.6}$$

Then we see that $\tilde{\mathcal{L}}(t) \in \Lambda + \Gamma(U, \tilde{\mathcal{C}})[[t]] \otimes \bar{\mathfrak{b}}$ and $\tilde{\mathcal{W}}_\infty(t) \in \Gamma(U, \tilde{\mathcal{C}})[[t]] \otimes \bar{N}$ from the results of Section 3. Let $\tilde{\mathcal{I}}$ be the sheaf of the ideal of $\tilde{\mathcal{C}}$ whose sections on U are generated by $\tilde{\mathcal{L}}_{i,j}$, $i - j \geq 2$. We consider the quotient sheaf $\tilde{\mathcal{C}}/\tilde{\mathcal{I}}$ on V. Let $\tilde{\rho}$ be the canonical projection from $\tilde{\mathcal{C}}$ to $\tilde{\mathcal{C}}/\tilde{\mathcal{I}}$. We can define the sheaves $\tilde{L}ax$, $\tilde{S}ol$, $\tilde{\mathcal{I}}$, $\tilde{\mathcal{C}}[[t]]$, $\tilde{\mathcal{C}}((t))$, $\tilde{\mathcal{C}}_\mu$, $\bar{\tilde{\mathcal{C}}}$ and $\exp\tilde{\mathcal{L}}$ from $\tilde{\mathcal{C}}$ as we did in Section 3 and this section. We extend $\tilde{\rho}$ to $\bar{\tilde{\mathcal{C}}}$, $\bar{\tilde{\mathcal{C}}}[[t]]$, $\mathcal{Q}(\bar{\tilde{\mathcal{C}}})$ and $\bar{\tilde{\mathcal{C}}}((t))$ as follows.

(1) For $f = \sum_{\mu=0}^{\infty} f_\mu \in \Gamma(U, \bar{\tilde{\mathcal{C}}})$, we define $\tilde{\rho}(f) = \sum_{\mu=0}^{\infty} \tilde{\rho}(f_\mu)$, where $f_\mu \in \Gamma(U, \tilde{\mathcal{C}}_\mu)$.

(2) For $g(t) = \sum_{\mu=0}^{\infty} g_\mu t^\mu \in \Gamma(U, \bar{\tilde{\mathcal{C}}})[[t]]$, we define $\tilde{\rho}(g(t)) = \sum_{\mu=0}^{\infty} \tilde{\rho}(g_\mu)t^\mu$, where $g_\mu \in \Gamma(U, \bar{\tilde{\mathcal{C}}})$.

(3) For $f/g \in \Gamma(U, \mathcal{Q}(\bar{\tilde{\mathcal{C}}}))$, we define $\tilde{\rho}(f/g) = \tilde{\rho}(f)/\tilde{\rho}(g)$, where $f, g \in \Gamma(U, \bar{\tilde{\mathcal{C}}})$.

(4) For $f(t)/g(t) \in \Gamma(U, \bar{\tilde{\mathcal{C}}})((t))$, we define $\tilde{\rho}(f(t)/g(t)) = \tilde{\rho}(f(t))/\tilde{\rho}(g(t))$, where $f(t), g(t) \in \Gamma(U, \bar{\tilde{\mathcal{C}}})[[t]]$.

We want to apply $\tilde{\rho}$ to both sides of (4.5). Since $\tilde{\mathcal{W}}_\infty(t) \in \Gamma(U, \tilde{\mathcal{C}})((t - t_0)) \otimes \bar{N}$ and $\chi_0 \in \Gamma(U, \tilde{\mathcal{C}}^{\bar{N}}) \otimes \mathrm{Mat}(n, \mathbf{C})$, we can apply $\tilde{\rho}$ to $\tilde{\mathcal{W}}_\infty(t)$ and χ_0. But $\mathcal{W}_0(t)$ belongs to $\Gamma(U, \mathcal{C})((t - t_0)) \otimes (B \cap \sigma B\sigma^{-1})$ and $\mathcal{W}_\infty$ belongs to $\Gamma(U, \mathcal{C}) \otimes \bar{N}$. Then we see that we cannot apply $\tilde{\rho}$ to $\mathcal{W}_0(t)$ and $\mathcal{W}_\infty$, at a glance. Fortunately we obtain the following proposition.

Proposition 4.5. *There exists a morphism from $\mathcal{C}$ to $Q(\bar{\tilde{\mathcal{C}}})$. Then the morphism $\tilde{\rho}$ can be applied to both sides of (4.5). $\tilde{\rho}(\mathcal{W}_0(t))$ and $\tilde{\rho}(\mathcal{W}_\infty)$ are well defined.*

Proof. By the definition of $\tilde{\mathcal{W}}_\infty(t)$, we have

$$\tilde{\mathcal{L}} = \mathcal{W}_\infty(0)\mathcal{L}\mathcal{W}_\infty(0)^{-1}. \tag{4.7}$$

We have $\tilde{\mathcal{L}}_{i,j} \in \Gamma(U, \mathcal{Q}(\exp(-t_0\mathcal{L}))) \otimes_{\mathbf{C}} \Gamma(U, \mathcal{C})$ from (4.7). Since $\exp\mathcal{L}$ is a subsheaf of C^ω, we see that $Q(\exp(-t_0\mathcal{L}))$ is a subsheaf of $\mathcal{Q}(C^\omega)$. Note that $\mathcal{Q}(C^\omega)$ is the sheaf of the meromorphic functions on V. From (4.7) we obtain the relation $\tilde{\mathcal{L}}_{i,j} = \gamma_{i,j}(U)(\mathcal{L}) \in \Gamma(U, \mathcal{Q}(\bar{\mathcal{C}}))$, $1 \leq j \leq i \leq n$. This $\gamma(U)$ defines the morphism μ from $\tilde{\mathcal{C}}$ to $\mathcal{Q}(\bar{\mathcal{C}})$ as follows. For any open set U of V, $\mu(U)$ is defined by

$$\mu(U)f(\tilde{\mathcal{L}}_{1,1}, \tilde{\mathcal{L}}_{1,2}, \ldots) = f(\gamma_{1,1}(U)(\mathcal{L}), \gamma_{1,2}(U)(\mathcal{L}), \ldots)$$

for any $f \in \Gamma(U, \tilde{\mathcal{C}})$. For $L \in V$, put $L_{i,j} = \mathcal{L}_{i,j}(L)$ and $\tilde{L}_{i,j} = \tilde{\mathcal{L}}_{i,j}(L)$. Let U be an open set of V and L be a point of U. Then $\gamma(U)$ is a meromorphic map on U. Since $\tilde{\mathcal{L}}_{i,j} = \gamma_{i,j}(U)(\mathcal{L})$, we have $\tilde{\mathcal{L}}_{i,j}(L) = \gamma_{i,j}(U)(\mathcal{L})(L)$ for $L \in U$. Then we have $\tilde{L}_{i,j} = \gamma_{i,j}(U)(L)$. Let L be a generic point of V and $U(L)$ be the neighbourhood of L contained by U sufficiently small that $\gamma(U)|_{U(L)} = \gamma(U(L))$ is an onto holomorphic map from $U(L)$ to $\gamma(U(L))(U(L))$. By the theorem of the inverse function, we see that $L_{i,j}$ can be expressed in terms of the analytic function of $\tilde{L}_{k,\ell}$, $1 \leq \ell \leq k \leq n$. This means that there exists a morphism from $\mathcal{C}$ to $\mathcal{Q}(\bar{\tilde{\mathcal{C}}})$. In other words, $\mathcal{L}_{i,j} \in \Gamma(U, \mathcal{Q}(\bar{\tilde{\mathcal{C}}}))$. Since $\mathcal{W}_0(t) \in \Gamma(U, \mathcal{C})((t - t_0)) \otimes (B \cap \sigma B\sigma^{-1})$ and $\mathcal{W}_\infty \in \Gamma(U, \mathcal{C}) \otimes \bar{N}$, we have $\mathcal{W}_0(t) \in \Gamma(U, \mathcal{Q}(\bar{\tilde{\mathcal{C}}}))((t - t_0)) \otimes (B \cap \sigma B\sigma^{-1})$ and $\mathcal{W}_\infty \in \Gamma(U, \mathcal{Q}(\bar{\tilde{\mathcal{C}}})) \otimes \bar{N}$. We can apply $\tilde{\rho}$ to $\mathcal{W}_0(t)$ and $\mathcal{W}_\infty$ by (1)–(4). □

Proposition 4.6. *Let*

$$\tilde{\mathcal{W}}_\infty(t)^{-1}\mathcal{W}_0(t) = \mathrm{e}^{(t-t_0)\chi_0}\mathcal{W}_\infty^{-1}\sigma^{-1}$$

be the Gauss decomposition of (4.5) and $\tilde{\rho}$ be the morphism defined above. Then (4.5) implies the Gauss decomposition

$$\tilde{\rho}(\tilde{\mathcal{W}}_\infty(t))^{-1}\tilde{\rho}(\mathcal{W}_0(t)) = \mathrm{e}^{(t-t_0)\tilde{\rho}(\chi_0)}\tilde{\rho}(\mathcal{W}_\infty)^{-1}\sigma^{-1}. \tag{4.8}$$

Furthermore (4.8) gives a section of the solution of the Toda lattice $\tilde{\rho}(\mathcal{W}_\infty(t))\tilde{\rho}(\chi_0)\tilde{\rho}(\mathcal{W}_\infty(t))^{-1}$ of the Jacobi element.

Proof. We see that $\tilde{\rho}(\mathcal{W}_\infty(t))\tilde{\rho}(\chi_0)\tilde{\rho}(\mathcal{W}_\infty(t))^{-1} = \tilde{\rho}(\tilde{\mathcal{L}}(t))$. We see that $\tilde{\rho}(\tilde{\mathcal{L}}(t))$ is a Lax operator of the Toda lattice from Proposition 3.7 of Section 3. □

Let us realize the Toda lattice associated with the small cell as orbits on the iso-level set of the full Kostant–Toda lattice. Put $m = (m_1, \dots, m_n) \in \mathbf{C}^n$. We define the algebraic variety S_m by $S_m := \{L \in V | \varphi_i(L) = m_i, \quad i = 1, \dots, n\}$. We call S_m the iso-level set of the full Kostant–Toda lattice of level m. The iso-level set S_m has the cell decomposition associated with $\mathcal{S}_n$. For $\sigma \in \mathcal{S}_n$, put

$$S_{m,\sigma} := \{L \in S_m | \mathrm{e}^{-t_0 L}\sigma^{-1} \in \bar{N}(B \cap \sigma B \sigma^{-1})\}.$$

Then we have

$$S_m = \sqcup_{\sigma \in \mathcal{S}_n} S_{m,\sigma}.$$

For $f(\tilde{\mathcal{L}}) \in \Gamma(U, \mathcal{Q}(\bar{\tilde{\mathcal{C}}}))$, we see that $\tilde{\rho}(f(\tilde{\mathcal{L}})) = f(\tilde{\rho}(\tilde{\mathcal{L}}))$ by definition. Suppose $f \in \Gamma(U, \mathcal{Q}(\bar{\tilde{\mathcal{C}}}))$ is a meromorphic function on V. Then $f(\tilde{\mathcal{L}})$ is an analytic function of $\tilde{L}$ in the neighbourhood of $\tilde{L}_0 \in U$, a generic point of V. Suppose $\tilde{L} \in V$ is a Jacobi element; then we see that $\rho(\tilde{\mathcal{L}})(\tilde{L}) = \tilde{L}$. Then we have $\tilde{\rho}(f(\tilde{\mathcal{L}}))(\tilde{L}) = f(\tilde{\mathcal{L}})(\tilde{L})$. Both sides of (4.5) and $\tilde{\mathcal{L}}(t) = \tilde{\mathcal{W}}_\infty(t)\chi_0\tilde{\mathcal{W}}_\infty(t)^{-1}$ are matrix valued analytic functions on V. Put $\tilde{L}(t) = \tilde{\mathcal{L}}(t)(\tilde{L})$, $\tilde{W}_\infty(t) = \tilde{\mathcal{W}}_\infty(t)(\tilde{L})$, $W_0(t) = \mathcal{W}_0(t)(\tilde{L})$ and $W_\infty = \mathcal{W}_\infty(\tilde{L})$. If $\tilde{L}$ is a Jacobi element, we have

$$\tilde{\rho}(\tilde{\mathcal{L}}(t))(\tilde{L}) = \tilde{L}(t), \quad \tilde{\rho}(\tilde{\mathcal{W}}_\infty(t))(\tilde{L}) = \tilde{W}_\infty(t),$$
$$\tilde{\rho}(\mathcal{W}_0(t))(\tilde{L}) = W_0(t) \quad \text{and} \quad \tilde{\rho}(\mathcal{W}_\infty)(\tilde{L}) = W_\infty.$$

Moreover since $\tilde{\rho}\varphi_i(\tilde{\mathcal{L}}) = \underline{\varphi_i}(\tilde{\rho}(\tilde{\mathcal{L}}))$, $i = 1, \dots, n$, we have $\tilde{\rho}(\chi_0)(\tilde{L}) = \chi_0(\tilde{L})$. Since $\mathcal{W}_\infty(0) \in \Gamma(U, \bar{\mathcal{C}}) \otimes \bar{N}$, we see that $\mathcal{W}_\infty(0) \in \Gamma(U, \mathcal{Q}(\bar{\tilde{\mathcal{C}}})) \otimes \bar{N}$. Then we can define $W_\infty(0) = \mathcal{W}_\infty(0)(\tilde{L})$ and $W_\infty(0) = \tilde{\rho}(\mathcal{W}_\infty(0))(\tilde{L})$ if $\tilde{L}$ is a Jacobi element. Then we have the following theorem.

Theorem 4.7. *Suppose* $L = W_\infty(0)^{-1}\tilde{L}W_\infty(0) \in S_{m,\sigma}$ *and* $\tilde{L}$ *is a Jacobi element. Then* $\tilde{L}(t)$ *is equal to* $\tilde{W}_\infty(t)\chi_0(m)\tilde{W}_\infty(t)^{-1}$, *where*

$$\chi_0(m) = \Lambda + \sum_{i=1}^{n} m_i E_{i,1},$$

and $\tilde{W}_\infty(t)$ *and* $W_0(t)$ *satisfy the Gauss decomposition*

$$\tilde{W}_\infty(t)^{-1}W_0(t) = \mathrm{e}^{(t-t_0)\chi_0(m)}W_\infty^{-1}\sigma^{-1}. \tag{4.9}$$

Moreover $\tilde{L}(t)$ *satisfies the Lax equation* $\frac{\mathrm{d}\tilde{L}(t)}{\mathrm{d}t} = [\tilde{L}(t)_+, \tilde{L}(t)]$ *and* $\tilde{L}(t)$ *is a Jacobi element in the neighbourhood of* $t = 0$.

The inverse of Theorem 4.7 holds.

Corollary 4.8. *Suppose* $L = W_\infty(0)^{-1}\tilde{L}W_\infty(0) \in S_{m,\sigma}$ *and* $\tilde{L}$ *is a Jacobi element. If* $U_\infty(t) \in \bar{N}$ *and* $U_0(t) \in B$ *satisfies the Gauss decomposition*

$$U_\infty(t)^{-1}U_0(t) = \mathrm{e}^{(t-t_0)\chi_0(m)}W_\infty^{-1}\sigma^{-1}, \tag{4.10}$$

then $\tilde{M}(t) = U_\infty(t)\chi_0(m)U_\infty(t)^{-1}$ *is a Lax operator of the Toda lattice and* $U_0(t) \in B \cap \sigma B\sigma^{-1}$.

Proof. Note that if $\tilde{L}$ is given, then W_∞^{-1} possesses the $\tilde{L}$ data in the decomposition (4.10). By the uniqueness of the Gauss decomposition, $U_\infty(t)$ and $U_0(t)$ coincide with $\tilde{W}_\infty(t)$ and $W_0(t)$ of Theorem 4.7 respectively. Theorem 4.7 guarantees that $\tilde{M}(t) = \tilde{W}_\infty(t)\chi_0(m)\tilde{W}_\infty(t)^{-1}$ is a Lax operator of the Toda lattice and $U_0(t) = W_0(t) \in B \cap \sigma B\sigma^{-1}$. □

Let us apply Corollary 4.8 to the case of $\sigma = id$. In this case the original Gauss decomposition is $W_\infty(t)^{-1}W_0(t) = \mathrm{e}^{(t-t_0)L}$. Since $W_\infty(t)$ does not have a pole at $t = t_0$ in general, we may put $t_0 = 0$ and we can consider $W_\infty(0)$. In this case we have $W_\infty(0) = 1_n$ by the uniqueness of the decomposition. If $\tilde{L} = W_\infty(0)LW_\infty(0)^{-1}$ satisfies the condition of Corollary 4.8, this means that $L \in S_{m,\phi}$ and $\tilde{L}$ is a Jacobi element. But in the case of $\sigma = id$, we see that $L = \tilde{L}$. This is nothing but the condition of Proposition 1.1. Consider the Gauss decomposition

$$\tilde{W}_\infty(t)^{-1} W_0(t) = \mathrm{e}^{t\chi_0(m)} W_\infty^{-1}. \tag{4.11}$$

But we have

$$W_\infty^{-1} W_\infty(t)^{-1} W_0(t) = \mathrm{e}^{t\chi_0(m)} W_\infty^{-1}$$
$$W_\infty(t)^{-1} W_0(t) = \mathrm{e}^{tL}.$$

This is also nothing but a condition of Proposition 1.1. We see that $\tilde{L}(t) = \tilde{W}_\infty(t)\chi_0(m)\tilde{W}_\infty(t)^{-1}$ is a Lax operator of the Toda lattice from Corollary 4.8. But we have

$$\tilde{L}(t) = W_\infty(t) W_\infty \chi_0(m) W_\infty^{-1} W_\infty(t)^{-1} = W_\infty(t) L W_\infty(t)^{-1}$$

and $\tilde{L}(0) = L$; these are nothing but the conclusions of Proposition 1.1.

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Appendix

In this case we consider $G = GL(2, \mathbf{R})$. Let V be the set of 2×2 Hessenberg matrices whose eigenvalues are all 0, that is,

$$V = \left\{ \begin{pmatrix} a & 1 \\ -a^2 & -a \end{pmatrix} \middle| a \in \mathbf{R} \right\}.$$

Put $S_0(t) := \{\mathrm{e}^{tL} | L \in V\}$, that is,

$$S_0(t) = \left\{ \begin{pmatrix} 1 + ta & t \\ -ta^2 & 1 - ta \end{pmatrix} \middle| a \in \mathbf{R} \right\}.$$

It is easy to see that $S_0(t)$ is homeomorphic to $\mathbf{R}$ for $t \neq 0$. Let $\bar{S}_0(t)$ be the compactification of $S_0(t)$. We regard $\bar{S}_0(t)$ as the circle of radius $|t|$. When t varies from 1 to 0, $\bar{S}_0(t)$ shrinks to one point $\bar{S}_0(0) = \{1_2\}$ along the cone (see Fig. A.1).

For $\mathrm{e}^{tL(a)} \in S_0(t)$, we consider the Gauss decomposition

$$W_\infty(t)^{-1} W_0(t) = \mathrm{e}^{tL(a)},$$

where

$$W_\infty(t) = \begin{pmatrix} 1 & 0 \\ w(t) & 1 \end{pmatrix} \quad \text{and} \quad W_0(t) = \begin{pmatrix} x(t) & y(t) \\ 0 & z(t) \end{pmatrix}.$$

We see that $w(t) = \frac{-ta^2}{1+ta}$. When $t = 1$, we cannot perform Gauss decomposition for $\mathrm{e}^{L(-1)} = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$. Let $P_1(-1)$ be the point of $\bar{S}_0(1)$ corresponding to $\mathrm{e}^{L(-1)}$. Then $\bar{S}_0(1)$ has a cell decomposition associated with the Bruhat decomposition $G/B = \bar{N}B/B \sqcup \bar{N}\sigma B/B$, where $\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, such as

$$\bar{S}_0(1) = (\bar{S}_0(1) - P_1(-1)) \sqcup P_1(-1).$$

In the same way, $\bar{S}_0(t)$ has the decomposition

$$\bar{S}_0(t) = (\bar{S}_0(t) - P_t(-1)) \sqcup P_t(-1),$$

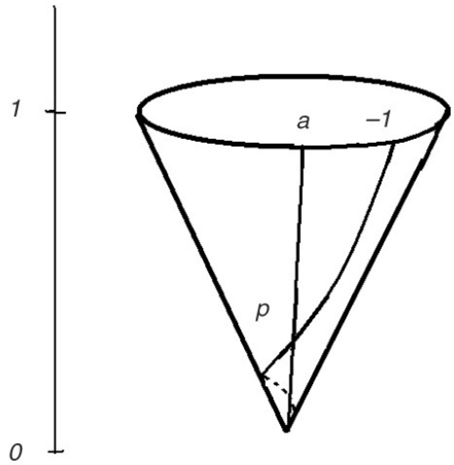

Fig. A.1.

where $P_t(-1)$ is the point of $\bar{S}_0(t)$ corresponding to $\begin{pmatrix} 0 & t \\ -1/t & 2 \end{pmatrix}$. When t varies from 1 to 0, $P_t(-1)$ moves along the curve on the cone of Fig. A.1. This curve is the flow of the Toda lattice whose initial point belongs to the σ-cell and that preserves the σ-cell. Let $Q_t(a)$ be the point of $\bar{S}_0(t)$ corresponding to $e^{tL(a)}$, where $a \neq -1$. $Q_t(a)$ moves along the line on the cone. Suppose the curve of $P_t(-1)$ and the line of $Q_t(a)$ cross at P (at $t = t_P$). This shows us that the Toda flow with initial point $L(a)$ of the ϕ-cell leaves the σ-cell and has a pole at $t = t_P$ [7,9,10].

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